LIBERTY PAPER SET

STD. 12 : Mathematics

Full Solution

Time : 3 Hours **ASSIGNMENT PAPER 13**

PART A

1. (B) **2.** (B) **3.** (B) **4.** (A) **5.** (C) **6.** (D) **7.** (A) **8.** (A) **9.** (B) **10.** (C) **11.** (C) **12.** (D) **13.** (D) **14.** (D) **15.** (A) **16.** (B) **17.** (A) **18.** (A) **19.** (C) **20.** (D) **21.** (B) **22.** (B) **23.** (A) **24.** (C) **25.** (A) **26.** (C) **27.** (B) **28.** (B) **29.** (B) **30.** (C) **31.** (C) **32.** (D) **33.** (B) **34.** (B) **35.** (D) **36.** (C) **37.** (B) **38.** (D) **39.** (B) **40.** (A) **41.** (A) **42.** (D) **43.** (B) **44.** (D) **45.** (D) **46.** (D) **47.** (D) **48.** (D) **49.** (A) **50.** (C)

PART B

SECTION A

1.

L.H.S. $= \left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

Suppose, Take $x = \cos 2\theta$

$\therefore\ 2\theta = \cos^{-1} x\ ,\quad 2\theta \in [0, \pi]$

$\therefore\ \theta = \dfrac{1}{2}\cos^{-1} x\ ,\quad \theta \in \left[0, \dfrac{\pi}{2}\right]$

$= \tan^{-1}\left[\dfrac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$

$= \tan^{-1}\left[\dfrac{1-\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}}{1+\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}}\right]$

$= \tan^{-1}\left[\dfrac{1-\sqrt{\tan^2\theta}}{1+\sqrt{\tan^2\theta}}\right]$

$= \tan^{-1}\left[\dfrac{1-|\tan\theta|}{1+|\tan\theta|}\right]$

$= \tan^{-1}\left[\dfrac{1-\tan\theta}{1+\tan\theta}\right]$

$\Bigg(\because\ \dfrac{-1}{\sqrt{2}} \le x \le 1$

$\therefore\ -\cos\dfrac{\pi}{4} \le \cos 2\theta \le \cos 0$

$\therefore\ \cos\left(\pi - \dfrac{\pi}{4}\right) \le \cos 2\theta \le \cos 0$

$\therefore\ \cos\dfrac{3\pi}{4} \le \cos 2\theta \le \cos 0$

$\therefore\ 0 \le 2\theta \le \dfrac{3\pi}{4}$ ($\because \cos\theta$ is decreasing function in 1st in 2nd quadrant.)

$\therefore\ 0 \le \theta \le \dfrac{3\pi}{8}$

$\therefore\ \tan\theta > 0$

$\therefore\ |\tan\theta| = \tan\theta \Bigg)$

$= \tan^{-1}\left[\dfrac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right]$

$= \tan^{-1}\left(\tan\left(\dfrac{\pi}{4}-\theta\right)\right)$

Here, $0 \le \theta \le \dfrac{3\pi}{8}$

$\Rightarrow -\dfrac{3\pi}{8} \le -\theta \le 0$

$\Rightarrow -\dfrac{\pi}{8} \le \dfrac{\pi}{4} - \theta \le \dfrac{\pi}{4}$

$\Rightarrow \left(\dfrac{\pi}{4}-\theta\right) \in \left[-\dfrac{\pi}{8}, \dfrac{\pi}{4}\right] \subset \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

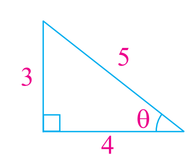
$= \dfrac{\pi}{4} - \theta$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x$

= R.H.S.

2.

L.H.S. $= 2\sin^{-1}\dfrac{3}{5}$



$\sin^{-1}\dfrac{3}{5} = \theta$

$\therefore \quad \sin\theta = \frac{3}{5}$

Here, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

Now, $2\sin\frac{3}{5} = 2\theta$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\left(\frac{3}{4}\right)}{1-\frac{9}{16}}$$

$$\therefore \quad \tan 2\theta = \frac{\frac{3}{2}}{\frac{7}{16}}$$

$$\therefore \quad \tan 2\theta = \frac{24}{7}$$

$$\therefore \quad 2\theta = \tan^{-1}\left(\frac{24}{7}\right)$$

$$\therefore \quad 2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$$

3.

Take long of both the sides,

$y = (\sin x - \cos x)^{(\sin x - \cos x)}$

$\therefore \log y = (\sin x - \cos x)\log(\sin x - \cos x)$

Now, take differentiation by x both the sides,

$$\frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\cdot\frac{1}{\sin x - \cos x}\cdot(\cos x + \sin x) + \log(\sin x - \cos x)(\cos x + \sin x)$$

$$= (\cos x + \sin x) + \log(\sin x - \cos x)\cdot(\cos x + \sin x)$$

$$\therefore \quad \frac{dy}{dx} = y\,[1 + \log(\sin x - \cos x)]\,(\cos x + \sin x)$$

$$\therefore \quad \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\,(1 + \log(\sin x - \cos x))(\cos x + \sin x)$$

4.

Method 1 :

$$I = \int \frac{\cos 2x}{(\cos x + \sin x)^2}\,dx$$

$$= \int \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2\sin x\cos x}\,dx$$

$$= \int \frac{\cos 2x}{1 + \sin 2x}\,dx$$

Here, $1 + \sin 2x = t$

$\therefore 2\cos 2x\,dx = dt$

$\therefore \cos 2x \cdot dx = \frac{dt}{2}$

$$\therefore I = \int \frac{1}{t}\,\frac{dt}{2}$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}\log|t| + c$$

$$\therefore I = \frac{1}{2}\log|1 + \sin 2x| + c$$

$$\therefore I = \frac{1}{2}\log|\cos^2 x + \sin^2 x + 2\sin x\cos x| + c$$

$$\therefore I = \frac{1}{2}\log|(\cos x + \sin x)^2| + c$$

$$\therefore I = \log|\sin x + \cos x| + c$$

Method 2 :

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2}\,dx$$

$$= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\,dx$$

$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2}\,dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

Here, $\cos x + \sin x = t$ ykËuþ ÷uíkkt,

$\therefore \quad (-\sin x + \cos x)\,dx = dt$

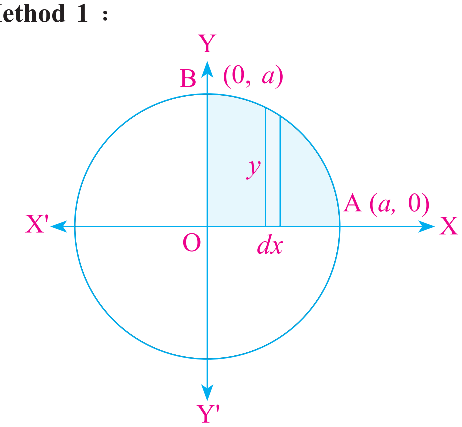
$\therefore \quad (\cos x - \sin x)\,dx = dt$

$$= \int \frac{dt}{t}$$

$$= \log|\cos x + \sin x| + c$$

5.

Method 1 :



From fig. the whole area enclosed by the given circle $= 4 \times$ (area of region AOBA bounded by the curve x-axis and ordinates $x = 0$ and $x = a$). (as the circle is symmetrical about both X-axis and Y-axis)

Required Area $= 4\int_0^a y\,dx$ (taking vertical strips)

$$= 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

Now, $x^2 + y^2 = a^2$ from $y = \pm\sqrt{a^2 - x^2}\, dx$ well get.

As the region AOBA lies in the first quadrant, $y = \sqrt{a^2 - x^2}$ is taken as positive. Integrating we get the whole area exclosed by the given circle.

$$\text{Required Area} = 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a}\right]_0^a$$

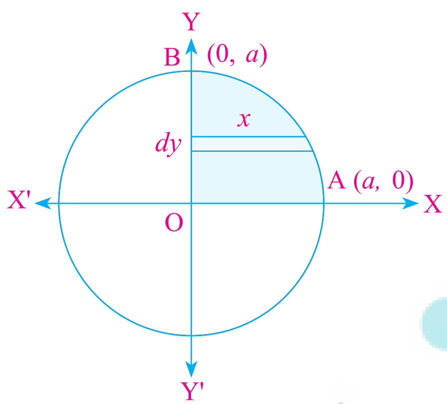
$$= 4\left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2} sin^{-1} 1\right) - 0\right]$$

$$= 4\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}\right)$$

$= \pi a^2$ sq. units.

Method 2 :

Considering horizontal strips as shown in fig., the whole area of the region enclosed by circle.



$$= 4\int_0^a x\, dy$$

$$= 4\int_0^a \sqrt{a^2 - y^2}\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2} sin^{-1}\frac{y}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2} sin^{-1} 1\right) - 0\right]$$

$$= 4\,\frac{a^2}{2}\,\frac{\pi}{2}$$

$= \pi a^2$ sq. units.

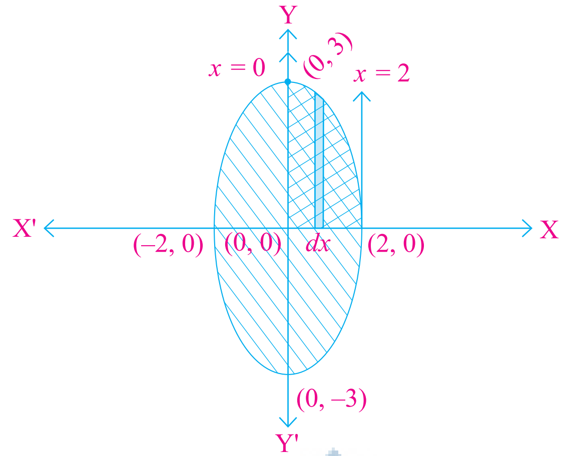
6.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$a^2 = 4,\ a = 2$

$b^2 = 9,\ b = 3$

$b > a$



Required Area :

A = 4 × Area bounded in the first quadrant.

$\therefore$ A = 4|I|

$$I = \int_0^2 y\, dx$$

nðu, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\therefore y^2 = 9\left[1 - \frac{x^2}{4}\right]$

$= \frac{9}{4}(4 - x^2)$

$\therefore y = \frac{3}{2}\sqrt{4 - x^2}$

$$I = \int_0^2 \frac{3}{2}\sqrt{4 - x^2}\, dx$$

$$I = \frac{3}{2}\int_0^2 \sqrt{4 - x^2}\, dx$$

$$= \frac{3}{2}\int_0^2 \sqrt{2^2 - x^2}\, dx$$

$$I = \frac{3}{2}\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2} sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$I = \frac{3}{2}\left[\left(\frac{2}{2}(0) + 2\, sin^{-1}(1)\right) - (0)\right]$$

$$I = \frac{3}{2} \cdot 2 \cdot \frac{\pi}{2}$$

$$I = \frac{3\pi}{2}$$

Now, A = 4|I|

$$= 4\left|\frac{3\pi}{2}\right|$$

$\therefore$ A = 6π sq. units.

7.

$$\frac{dy}{dx} = \frac{1 - cos\, x}{1 + cos\, x}$$

$$\therefore dy = \left(\frac{1 - cos\, x}{1 + cos\, x}\right) dx$$

$$\therefore dy = tan^2\frac{x}{2}\, dx$$

→ Integrate both sides,

$$\therefore \int dy = \int \tan^2 \frac{x}{2}\, dx$$

$$\therefore \int dy = \int \left(\sec^2 \frac{x}{2} - 1\right) dx$$

$$\therefore \quad y = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + c$$

$$\therefore \quad y = 2 \tan \frac{x}{2} - x + c;$$

Which is required general solution of given differential equation.

8.

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Now, Take $\vec{x} = \vec{a} + \vec{b}$

$\therefore \quad \vec{x} = 4\hat{i} + 4\hat{j} + 0\hat{k}$

$\vec{y} = \vec{a} - \vec{b}$

$\therefore \quad \vec{y} = 2\hat{i} + 0\hat{j} + 4\hat{k}$

Unit vector perpendicular to each of the vector

$$\vec{x} \text{ and } \vec{y} = \pm \frac{\vec{x} \times \vec{y}}{|\vec{x} \times \vec{y}|}$$

Now, $\vec{x} \times \vec{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

$\vec{x} \times \vec{y} = 16\hat{i} - 16\hat{j} - 8\hat{k}$

$|\vec{x} \times \vec{y}| = \sqrt{256 + 256 + 64}$

$= \sqrt{576}$

$= 24$

Unit vector perpendicular to each the vector $\vec{x}$ and $\vec{y}$

$$= \pm \frac{(16\hat{i} - 16\hat{j} - 8\hat{k})}{24}$$

$$= \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

9.

Here, $\vec{b_1} = \hat{i} - \hat{j} - 2\hat{k};$

$\vec{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$

Suppose, Angle between two line is α then,

$$\therefore \cos\alpha = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|} \quad \text{......... (1)}$$

$\vec{b_1} \cdot \vec{b_2} = (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k})$

$= 3 + 5 + 8$

$= 16$

$|\vec{b_1}| = \sqrt{1 + 1 + 4}$

$= \sqrt{6}$

$|\vec{b_2}| = \sqrt{9 + 25 + 16}$

$= \sqrt{50}$

$$\therefore \cos\alpha = \frac{|16|}{\sqrt{50}\sqrt{6}}$$

$$\therefore \cos\alpha = \frac{|16|}{5\sqrt{2}\sqrt{6}}$$

$$= \frac{16}{5\sqrt{12}}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{16}{5\sqrt{12}}\right)$$

$$\therefore \alpha = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Therefore, the angle between two line is $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$.

10.

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$

Direction of line $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

Vector equation of line,

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\therefore \vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}), \lambda \in R$

Cartesian equation : $\frac{x-2}{1} = \frac{y+1}{2} = \frac{x-4}{-1}$

11.

We know that the sample space is,

S = {1, 2, 3, 4, 5, 6}

Now E = {3, 6}, F = {2, 4, 6} and E ∩ F = {6}.

Then $P(E) = \frac{2}{6} = \frac{1}{3}$,

$P(F) = \frac{3}{6} = \frac{1}{2}$ and

$P(E \cap F) = \frac{1}{6}$

Clearly P(E ∩ F) = P(E) · P(F)

Hence E and F are independent events.

12.

Method 1 :

We have, P(at least one of A and B)

= P(A ∪ B)

= P(A) + P(B) – P(A ∩ B)

= P(A) + P(B) – P(A) P(B)

= P(A) + P(B) (1 – P(A))

= P(A) + P(B) P(A')

= 1 – P(A') + P(B) P(A')

= 1 – P(A') [1 – P(B)]

= 1 – P(A') P(B')

Method 2 :

$P(A \cup B) = 1 - P((A \cup B)')$

$= 1 - P(A' \cap B')$

$= 1 - P(A')\, P(B')$

(Because A', B' is independent)

SECTION B

13.

$\forall\ x_1, x_2 \in R$ (Domain) $\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow 3 - 4x_1 = 3 - 4x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one function.

Suppose, $y \in R$ (co-domain) $y = f(x)$

$\therefore\ y = 3 - 4x$

$\therefore\ 4x = 3 - y$

$\therefore\ x = \frac{3-y}{4} \in R$

Now, $f(x) = f\left(\frac{3-y}{4}\right)$

$= 3 - 4\left(\frac{3-y}{4}\right)$

$= 3 - 3 + y$

$= y$

Thus for every $y \in R$ (co-domain) $x = \frac{3-y}{4} \in R$ such that $f(x) = y$.

$\therefore$ f is onto function.

14.

Here, $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

Now, $A + A^T = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore\ \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now, $A - A^T = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$

$\therefore\ \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$= A$

15.

$A_{11} = -6,\ A_{12} = 4,\ A_{21} = -3,\ A_{22} = 2$

$adj\ A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$A\,(adj\ A) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .... (1)

$(adj\ A)\,A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .... (2)

$|A| = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$= -12 + 12$

$= 0$

$|A|\, I_2 = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .... (3)

From equation (1), (2) and (3),

$A(adj\ A) = (adj\ A)\,A = |A|\, I$

16.

$e^y (x + 1) = 1$ Lkwt

Differentiate w.r.t. x,

$e^y (1) + (x + 1)\, e^y \frac{dy}{dx} = 0$

$\therefore\ (x + 1)\, e^y \frac{dy}{dx} = -e^y$

$\therefore \quad \frac{dy}{dx} = \frac{-1}{(x+1)}$ .... (1)

Differentiate with respect to x,

$$\frac{d^2y}{dx^2} = -\left(\frac{-1}{(x+1)^2}\right)\frac{d}{dx}(x+1)$$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{(1+x)^2}$ .... (2)

But, $\left(\frac{dy}{dx}\right)^2 = \frac{1}{(x+1)^2}$ .... (3)

$\therefore$ From equation (2) and (3),

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

17.

$f'(x) = \cos x + \sin x$

$f''(x) = -\sin x + \cos x$

For finding maxima and minimum of $f(x)$.

$f'(x) = 0$

$\therefore \quad \cos x + \sin x = 0$

$\therefore \quad \cos x = -\sin x$

$\therefore \quad \tan x = -1$

$\therefore \quad x = \frac{3\pi}{4}, \frac{7\pi}{4} \qquad (\because 0 < x < 2\pi)$

$\rightarrow$ For $x = \frac{3\pi}{4}$,

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} \qquad \left(\begin{array}{l}\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}\\ \cos\frac{3\pi}{4} = \frac{-1}{\sqrt{2}}\end{array}\right)$$

$$= \frac{-1}{\sqrt{2}} \; \frac{-1}{\sqrt{2}}$$

$$= -\sqrt{2} < 0$$

$\therefore$ f has local maximum value at $x = \frac{3\pi}{4}$.

$\therefore \quad f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$

$\therefore \quad f\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \sqrt{2}$

$\therefore$ Local maximum value $= \sqrt{2}$

$\rightarrow$ For $x = \frac{7\pi}{4}$,

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} \qquad \left(\begin{array}{l}\sin\frac{7\pi}{4} = \frac{-1}{\sqrt{2}}\\ \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}\end{array}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \sqrt{2} > 0$$

$\therefore$ f has local minimum value at $x = \frac{7\pi}{4}$.

$$f\left[\frac{7\pi}{4}\right] = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$

$$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\sqrt{2}$$

$\therefore$ Local minimum value $= -\sqrt{2}$

18.

Let $\vec{\beta_1} = \lambda\vec{\alpha}$ λ is a scalar,

i.e., $\vec{\beta_1} = 3\lambda\vec{i} - \lambda\vec{j}$

Now, $\vec{\beta_2} = \vec{\beta} - \vec{\beta_1}$

$= (2-3\lambda)\hat{i} + (1+\lambda)\hat{j} - 3\hat{k}$

Now, since $\vec{\beta_2}$ is to be perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}\cdot\vec{\beta_2} = 0$ i.e.,

$3(2-3\lambda) - (1+\lambda) = 0$

or, $\lambda = \frac{1}{2}$.

Therefore, $\vec{\beta_1} = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$ and $\vec{\beta_2} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

19.

L : $\vec{r} = (6\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

M : ($\vec{r} = -4\hat{i} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

$\therefore \quad \vec{a_1} = 6\hat{i} + 2\hat{j} + 2\hat{k}$; and

$\vec{b_1} = \hat{i} - 2\hat{j} + 2\hat{k}$

and $\vec{a_2} = -4\hat{i} - \hat{k}$; and

$\vec{b_2} = 3\hat{i} - 2\hat{j} - 2\hat{k}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$

$= 8\hat{i} + 8\hat{j} + 4\hat{k}$

$\vec{b_1} \times \vec{b_2} \neq \vec{0}$

$\therefore$ Lines are intersecting or skew lines.

$|\vec{b_1} \times \vec{b_2}| = \sqrt{64+64+16}$

$= \sqrt{144}$

$= 12$

$\vec{a_2} - \vec{a_1} = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k})$

$= -10\hat{i} - 2\hat{j} - 3\hat{k}$

$(\vec{a_2} - \vec{a_1})\cdot(\vec{b_1} \times \vec{b_2})$

$= (-10\hat{i} - 2\hat{j} - 3\hat{k})\cdot(8\hat{i} + 8\hat{j} + 4\hat{k})$

$= -80 - 16 - 12$

$= -108$

$\neq 0$

$\therefore$ Lines are skew lines

Shortest distance between skew lines,

$$= \frac{\left|(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})\right|}{\left|\vec{b_1}\times\vec{b_2}\right|}$$

$$= \frac{|-108|}{12}$$

$$= \frac{108}{12}$$

$= 9$ unit

20.

$x \geq 0,\ y \geq 0$

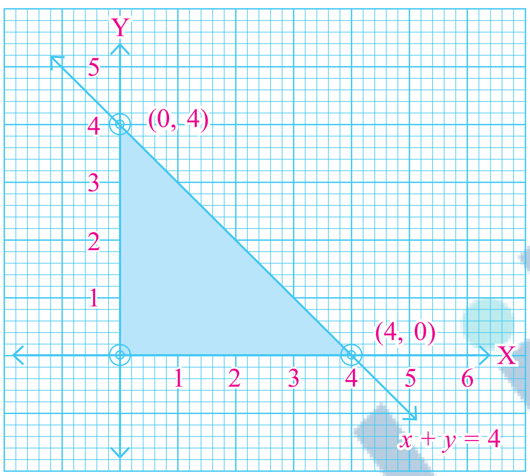
objective function $Z = 3x + 4y$

$x + y = 4$

x	0	4
y	4	0

(0, 4) $x \geq 0,\ y \geq 0$

(4, 0) (0, 0)



The shaded region in fig. s feasible region determined by the system of constraints which is bounded. The co-ordinates of corner points are (0, 0), (4, 0) and (0, 4).

Corner Point	Corresponding value of $Z = 3x + 4y$
(4, 0)	$Z = 12$
(0, 4)	$Z = 16 \leftarrow$ Maximum
(0, 0)	$Z = 0$

Thus, Maximum value of Z is 10 at point (0, 4).

21.

Event E_1 : Student knows the answer

Event E_2 : Student guesses the answer

$$P(E_1) = \frac{3}{4}$$

$$P(E_2) = \frac{1}{4}$$

Event A : Student gives answer correctly

Probability that the student knows the answer given that he answered it correctly

$$P(E_1 \mid A) = \frac{P(E_1)\cdot P(A \mid E_1)}{P(A)}$$

$$\therefore\ P(A) = P(E_1)\cdot P(A \mid E_1) + P(E_2)\cdot P(A \mid E_2)$$

$$= \frac{3}{4}\times 1 + \frac{1}{4}\times\frac{1}{4}$$

$$= \frac{3}{4} + \frac{1}{16}$$

$$= \frac{13}{16}$$

$$\therefore\ P(E_1 \mid A) = \frac{\frac{3}{4}\times 1}{\frac{13}{18}}$$

$$= \frac{12}{13}$$

SECTION C

22.

Here, $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Let, $P = \frac{1}{2}(B + B')$

$$= \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix}$$

Now, $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus, $P = \frac{1}{2}(B + B')$ is Symmetric matrix.

Also, Let

$$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then, $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{2} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q.$

Thus, $Q = \frac{1}{2}(B - B')$ is Skew Symmetric matrix

Now, $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

23.

The equation can be represented as matrix form,

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$\therefore$ AX = B

Where, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

AX = B

$\therefore X = A^{-1}B$

For finding A^{-1},

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix}$$

$= 2(10 + 3) - 1(-5 + 0) + 1(3 - 0)$

$= 2(13) + 5 + 3$

$= 26 + 5 + 3$

$= 34 \neq 0$

We get Unique solution.

For finding *adj* A,

Co-factor of element 2 $A_{11} = (-1)^2 \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix}$

$= 1(10 + 3)$

$= 13$

Co-factor of element 1 $A_{12} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix}$

$= (-1)(-5 + 0)$

$= 5$

Co-factor of element 1 $A_{13} = (-1)^4 \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$

$= 1(3 - 0)$

$= 3$

Co-factor of element 1 $A_{21} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix}$

$= (-1)(-5 - 3)$

$= 8$

Co-factor of element −2 $A_{22} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix}$

$= 1(-10 + 0)$

$= -10$

Co-factor of element −1 $A_{23} = (-1)^5 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$

$= (-1)(6 - 0)$

$= -6$

Co-factor of element 0 $A_{31} = (-1)^4 \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}$

$= 1(-1 + 2)$

$= 1$

Co-factor of element 3 $A_{32} = (-1)^5 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$

$= (-1)(-2 - 1)$

$= 3$

Co-factor of element −5 $A_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$

$= 1(-4 - 1)$

$= -5$

$$adj\ A = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$= \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$= \frac{1}{34} \begin{bmatrix} 13 + 12 + 9 \\ 5 - 15 + 27 \\ 3 - 9 - 45 \end{bmatrix}$$

$$= \frac{1}{34} \begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Solution : $x = 1,\ y = \frac{1}{2},\ z = \frac{-3}{2}$

24.

Suppose, $u = \left(x + \frac{1}{x}\right)^x$ and $v = x^{\left(1 + \frac{1}{x}\right)}$

$\therefore y = u + v$

Now, differentiate w.r.t. x,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{......... (1)}$$

Here, $u = \left(x + \frac{1}{x}\right)^x$

Take *log* both the sides,

$$log\ u = x\ log\left(x + \frac{1}{x}\right)$$

Now, differentiate w.r.t. x,

$$\therefore \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\ log\left(x + \frac{1}{x}\right) + log\left(x + \frac{1}{x}\right)\ \frac{d}{dx}\ x$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{x}{\left(x+\frac{1}{x}\right)}\frac{d}{dx}\left(x+\frac{1}{x}\right) + log\left(x+\frac{1}{x}\right)$$

$$= \frac{x^2}{x^2+1}\left(1-\frac{1}{x^2}\right) + log\left(x+\frac{1}{x}\right)$$

$$= \frac{x^2}{x^2+1}\left(\frac{x^2-1}{x^2}\right) + log\left(x+\frac{1}{x}\right)$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{x^2-1}{x^2+1} + log\left(x+\frac{1}{x}\right)$$

$$\therefore \quad \frac{du}{dx} = u\left(\frac{x^2-1}{x^2+1} + log\left(x+\frac{1}{x}\right)\right)$$

$$\therefore \quad \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + log\left(x+\frac{1}{x}\right)\right] \quad \text{... (2)}$$

Now, $v = x^{\left(1+\frac{1}{x}\right)}$

Take *log* both the sides,

$$log\ v = \left(1+\frac{1}{x}\right)\ log\ x$$

Now, differentiate w.r.t. x,

$$\frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right)\frac{d}{dx}\ logx + logx\ \frac{d}{dx}\left(1+\frac{1}{x}\right)$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right)\frac{1}{x} + log\,x\left(0-\frac{1}{x^2}\right)$$

$$= \frac{x+1}{x^2} - \frac{log\,x}{x^2}$$

$$\therefore \quad \frac{dv}{dx} = v\left(\frac{x+1-logx}{x^2}\right)$$

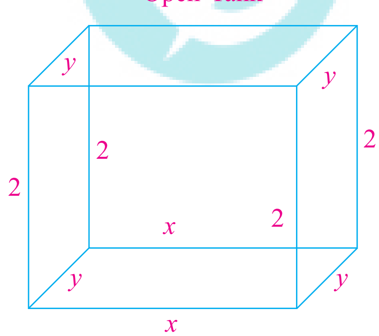
$$\therefore \quad \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-logx}{x^2}\right) \quad \text{...... (3)}$$

Put, the value of equation (2) and (3) in equation (1),

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-logx}{x^2}\right]$$

25.

Open Tank



Suppose, length of base with rectangular is x meter, breadth is y meter and height is 2 meter

∴ Volume of tank = 8 (meter)^3

∴ $x \times y \times 2 = 8$

∴ $xy = 4$ .... (1)

Now, Area of base Δ_1 = Lenght × Breadth

= xy

Δ_1 = 4 (meter)^2 or sq. meter

1 sq. meter = ₹ 70

∴ 4 sq. meter = (?) $= \frac{70 \times 4}{1}$ = ₹ 280

Area of four sides,

$\Delta_2 = 4y + 4x$

$\Delta_2 = 4(x + y)$ sq. meter

Cost of four sides = $45 \times 4(x + y)$

= $180\ (x + y)$

∴ Total cost = $280 + 180\ (x + y)$ .... (2)

$f(x) = 280 + 180(x + y)$

$$\therefore \quad f(x) = 280 + 180\left(x+\frac{4}{x}\right)$$

$$\therefore \quad f'(x) = 180\left(1-\frac{4}{x^2}\right)$$

$$\therefore \quad f''(x) = 180\left(\frac{8}{x^3}\right) > 0$$

→ For minimum cost,

$f'(x) = 0$

$$\therefore \quad 180\left(1-\frac{4}{x^2}\right) = 0$$

$$\therefore \quad 1 - \frac{4}{x^2} = 0$$

$$\therefore \quad 1 = \frac{4}{x^2}$$

∴ $x^2 = 4$

∴ $x = 2$ $\quad (\because x > 0)$

→ If $x = 2$ then,

$$\therefore \quad y = \frac{4}{x}$$

$$= \frac{4}{2}$$

∴ $y = 2$

→ From equation (2),

∴ Total cost = $280 + 180(2 + 2)$

= $280 + 720$

∴ Total cost = ₹ 1,000

26.

$$\text{Let I} = \int_0^{\frac{\pi}{2}} log(sin\ x)\ dx \quad \text{.... (1)}$$

By property (6),

$$\text{I} = \int_0^{\frac{\pi}{2}} log\ sin\left(\frac{\pi}{2}-x\right)\ dx$$

$$= \int_0^{\frac{\pi}{2}} log\ (cos\ x)\ dx \quad \text{.... (2)}$$

$\rightarrow$ Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cdot \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log (\sin x \cdot \cos x) + \log 2 - \log 2)\, dx$$

(by adding and subtracting $\log 2$)

$$= \int_0^{\frac{\pi}{2}} (\log (\sin 2x)\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$$

Put $2x = t$ in the first integral. Then $2dx = dt$,

when, $x = 0 \Rightarrow t = 0$ and

$x = \frac{\pi}{2} \Rightarrow t = \pi.$

$$\therefore 2I = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt - \log 2 \int_0^{\frac{\pi}{2}} 1\, dx$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log (\sin t)\, dt - \frac{\pi}{2} \log 2$$

(By property (7), $\sin(\pi - t) = \sin t$)

$$= \int_0^{\frac{\pi}{2}} \log (\sin x)\, dx - \frac{\pi}{2} \log 2$$

(by changing variable t to x)

$$2I = I - \frac{\pi}{2} \log 2$$

$$I = -\frac{\pi}{2} \log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log \sin x\, dx = -\frac{\pi}{2} \log 2$

27.

$$(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$$

$$\therefore \frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1}$$

$$\therefore \frac{e^y\, dy}{2 - e^y} = \frac{dx}{x+1}$$

$\rightarrow$ Integrate both the sides,

$$\therefore \int \frac{e^y\, dy}{(2 - e^y)} = \int \frac{dx}{x+1}$$

$$\therefore -\int -\frac{e^y\, dy}{(2 - e^y)} = \int \frac{dx}{x+1}$$

$$\therefore -\log |2 - e^y| = \log|x + 1| + \log|c|$$

$$\therefore \log \left|\frac{1}{2 - e^y}\right| = \log|c(x + 1)|$$

$$\therefore \frac{1}{2 - e^y} = c(x + 1) \quad \text{.... (1)}$$

$\rightarrow$ $y = 0$ when $x = 0$

$$\therefore \frac{1}{2 - e^0} = c(0 + 1)$$

$$\therefore \frac{1}{2 - 1} = c$$

$$\therefore c = 1$$

$\rightarrow$ Put the value of c in equation (1),

$$\therefore \frac{1}{2 - e^y} = (x + 1)$$

$$\therefore (2 - e^y)(x + 1) = 1$$

$$\therefore 2 - e^y = \frac{1}{x+1}$$

$$\therefore 2 - \frac{1}{x+1} = e^y$$

$$\therefore \frac{2x + 2 - 1}{x+1} = e^y$$

$$\therefore \frac{2x + 1}{x+1} = e^y$$

$$\therefore y = \log\left|\frac{2x+1}{x+1}\right|, x \neq -1,$$

which is required particular solution of given differential equation.